The Relativistic Hamilton-Jacobi Equation for a Massive, Charged and Spinning Particle, its Equivalent Dirac Equation and the de Broglie-Bohm Theory

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Abstract

Using the Clifford and the Spin-Clifford formalisms we prove that the classical relativistic Hamilton Jacobi equation for a charged massive (and spinning) particle interacting with an external electromagnetic field is equivalent to a Dirac-Hestenes equation satisfied by a class of spinor fields that we call classical spinor fields, characterized for having the Takabayashi angle function constant (equal to 0 or π). We also investigate a nonlinear Dirac-Hestenes like equation that comes from some class of generalized classical spinor fields. Finally we show that the general Dirac-Hestenes equation (which is a representative in the Clifford bundle of the usual Dirac equation) gives a generalized Hamilton-Jacobi equation where the quantum potential satisfy a severe constraint and the "mass of the particle" becomes a variable. Our results can then explain the experimental discrepancies found between prediction for the de Broglie-Bohm theory and recent experiments and it explains also theoretical found between de Broglie-Bohm formalism and quantum mechanics. We briefly discuss also the de Broglie's double solution theory in view of our results showing that it can be realized, at least in the case of spinning free particles..The paper contains several Appendices where notation and proofs of some results of the text are presented.

1 Introduction

In this paper we prove that the relativistic Hamilton-Jacobi equation for a massive particle charged particle moving in Minkowski spacetime and interacting with an electromagnetic field is equivalent to a Dirac equation satisfied by a spe-

cial class of Dirac spinor fields[1] (characterized for having the Takabayashi angle function equal to 0 or π). Also, any Dirac equation satisfied by these classical spinor fields implies in a corresponding relativistic Hamilton-Jacobi equation.

After proving these result we show that since general spinor fields which are solutions of the Dirac equation in an external potential have in general Takabayashi angle functions [1, 6] which are not constant like 0 orπ) the corresponding derived generalized Hamilton-Jacobi equation (GHJE) besides having a quantum potential which must satisfy a very severe constraint (see Eq.(29)) has also a variable mass (which is a function of the Takabayashi angle function). So, the usual equations derived from Schrödinger equation used in simulations of, e.g., the double slit experiments do not take into account that the mass of the particle which comes from the generalized HJE becomes a variable. This eventually must explain the discrepancies found between theoretical predictions from the de Broglie and Bohm formalism [7, 2, 22] and some results of experiments and found inconsistencies (such as necessity of surreal trajectories) as related, e.g., in [3, 4, 10, 23, 27, 36]. We briefly discuss also the de Broglie's double solution theory in view of our results, founding that, at least for the case of a free (spinning particle) it can be realized.

To show the above results we will use the Clifford bundle formalism where Dirac spinor fields are represented by an equivalence class of even sections of the Clifford bundle $\mathcal{C}\ell(M,\eta)$ of differential forms. Details about this theory and notations used may be found in [32, 31, 34] and a resume is given in the Appendix. Here we recall that in this paper all calculations are done in the Minkowski spacetime structure $(M \simeq \mathbb{R}^4, \boldsymbol{\eta}, D, \tau_\eta)$.

2 A Trivial Derivation of the Relativistic Hamilton-Jacobi Equation (HJE)

Let $\sigma : \mathbb{R} \to M,\ s \mapsto \sigma(s)$ be a timelike curve in spacetime time representing the motion of a particle of mass m and electrical charge e interacting with an electromagnetic field $F = dA$, where the potential $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ and $F \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$. Let $v = \boldsymbol{\eta}(\sigma_*, \ )\ \sigma_*$,be the velocity of the particle and define

$$v = \boldsymbol{\eta}(\sigma_*, \ ) \tag{1}$$

as a 1-form field over σ. Then, the motion of such a particle, as well known is governed in classical electrodynamics by the Lorentz force law, i.e.,

$$m\dot{v} = ev \lrcorner F. \tag{2}$$

where $\dot{v} = dv/ds$ Now, let $V \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ be a vector field such that

$$V|_\sigma = \dot{v}, \quad V^2 = 1 \tag{3}$$

[1] These special class of spinor fields will be called classical spinor fields.

As defined in the Appendix let $\{x^\mu\}$ be global coordinates for M in Einstein-Lorentz-Poincaré gauge, with $\{\gamma^\mu = dx^\mu\}$ be a basis for $\bigwedge^1 T^*M$ and moreover let us consider that[2] $\gamma^\mu \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$. In these coordinates the Dirac operator $\boldsymbol{\partial}$ and the representative of the spin-Clifford operator $\boldsymbol{\partial}^{(s)}$ acting on a representative $\psi \in \sec \mathcal{C}\ell(M,\eta)$ of Dirac-Hestenes spinor field $\Psi \in P_{\mathrm{spin}^0 1,3}(M,\eta) \times_\rho \mathbb{C}^4$ in a given spin frame $\Xi \in \sec P_{\mathrm{spin}^0 1,3}(M,\mathrm{g})$ are represented by $\gamma^\mu \partial_\mu$, i.e.,

$$\boldsymbol{\partial} = \gamma^\mu \partial_\mu \ , \quad \boldsymbol{\partial}^{(s)} = \gamma^\mu \partial_\mu. \tag{4}$$

Now, we can show the following identity[3]

$$\dot{v} = V \lrcorner (\boldsymbol{\partial} V)|_\sigma = V \lrcorner (\boldsymbol{\partial} \wedge V)|_\sigma = V \lrcorner (dV)|_\sigma \tag{5}$$

and thus we can write Eq.(2) as

$$V \lrcorner [d(mV - eA)]|_\sigma = 0 \tag{6}$$

In what follows we suppose that Eq.(6) holds for each integral line of the vector field $\mathbf{V} = \eta(V, \ )$, i.e.,

$$V \lrcorner [d(mV - eA)] = 0 \tag{7}$$

A sufficient condition for the validity of Eq.(7) is, of course the existence of a scalar function S such that

$$mV - eA = -dS = -\boldsymbol{\partial} S. \tag{8}$$

We immediately recognize the $\Pi := -\boldsymbol{\partial} S$ as the canonical momentum and of course taking into account that $V^2 = 1$ we get from Eq.(8) that

$$(\Pi + eA)^2 = m^2 \tag{9}$$

which is the relativistic Hamilton Jacobi equation [24].

3 A Classical Dirac-Hestenes Equation

To proceed, we recall that it is always possible to choice a gauge for the potential such that the components Π_μ of the canonical momentum are constants. We suppose in what follows that the potentials are already in this gauge. Moreover, we recall an invertible representative ψ (in the Clifford bundle) of Dirac-Hestenes spinor field can be written as[4]

$$\psi = \rho^{1/2} e^{\frac{\gamma^5 \beta}{2}} \boldsymbol{R} \in \sec(\bigwedge^0 T^*M + \bigwedge^2 T^*M + \bigwedge^4 T^*M) \hookrightarrow \sec \mathcal{C}\ell(M,\eta) \tag{10}$$

[2]Thus the 1-forms γ^μ satisfies $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$.

[3]Observe that identity given by Eq.(5) is valid in a general Lorentzian manifold.

[4]Recall that $\gamma^5 = \tau_{\boldsymbol{g}} \in \sec \bigwedge^4 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathrm{g})$ is the volume element 4-form field.

where $\rho, \beta \in \sec \bigwedge^0 T^*M$ and for each $x \in M$, $\boldsymbol{R}(x) \in \mathrm{Spin}^0_{1,3} \simeq \mathrm{Sl}(2,\mathbb{C})$ and $\boldsymbol{R}\boldsymbol{R}^{-1} = \boldsymbol{R}^{-1}\boldsymbol{R} = 1$ and $\boldsymbol{R} = \tilde{\boldsymbol{R}}$ is called a rotor. Of course. $\psi = \boldsymbol{R}^{-1}\rho^{-1/2}e^{-\frac{\gamma^5\beta}{2}}$.

Next choice a ψ such that

$$V = \psi\gamma^0\psi^{-1} = e^{\gamma^5\beta}\boldsymbol{R}\gamma^0\boldsymbol{R}^{-1}. \tag{11}$$

Since $V \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ we necessarily must have that the Takabayashi angle β must be 0 or π. As we said in the introduction we will call spinor fields satisfying this condition classical spinor fields. We now write $\boldsymbol{R} = R(\Pi)\ e^{S\gamma^{21}}$ where for each $x \in M$, $R(\Pi) \in \mathrm{Spin}^0_{1,3}$ is a rotor depending on Π. We then return to Eq.(8) and multiply it on the right with ψ getting

$$\Pi\psi = -\boldsymbol{\partial}S\psi = m\psi\gamma^0 - eA\psi = 0. \tag{12}$$

We now ask: is it possible to define a differential operator $\hat{\boldsymbol{\Pi}}$ acting on sections of the Clifford bundle such that

$$\hat{\boldsymbol{\Pi}}\psi = \Pi\psi \tag{13}$$

The answer is yes if we define

$$\hat{\boldsymbol{\Pi}}\psi = \boldsymbol{\partial}\psi\gamma^{21} \tag{14}$$

and use the particular classical spinor field

$$\psi = R(\Pi)e^{S\gamma^{21}}. \tag{15}$$

Indeed, in this case it is

$$\boldsymbol{\partial}\psi\gamma^{21} = -\boldsymbol{\partial}S\psi = \Pi\psi. \tag{16}$$

So, the classical spinor field given by Eq.(15) satisfies the first order partial equation[5]

$$\boldsymbol{\partial}\psi\gamma^{21} - m\psi\gamma^0 + eA\psi = 0. \tag{17}$$

Remark 1 Eq.(17) *for a general Dirac-Hestenes spinor field is known as the Dirac-Hestenes equation* [15] *which is a representative in the Clifford bundle* (*see Appendix C*) *of the traditional Dirac equation for a covariant Dirac spinor field* $\Psi \in \sec P_{\mathrm{spin}^0 1,3}(M,\eta) \times_\rho \mathbb{C}^4$ *which read in a chart for M with coordinates in in Einstein-Lorentz-Poincaré gauge*[6]

$$i\boldsymbol{\gamma}^\mu(\partial_\mu - ieA_\mu)\Psi + m\Psi = 0. \tag{18}$$

[5] Take notice that weare using a naturalsystem of unitis where the numerical values of Planck constant $\hbar$ and the speed of light c are equal to one.

[6] In Eq.(18) the $\boldsymbol{\gamma}^\mu$ are the Dirac matrices in standard representation.

So, we have proved that there is a class of classical spinor fields (the ones satisfying Eq.(15) such that the relativistic Hamilton-Jacobi equation is equivalent to the celebrate Dirac equation. Of course, it is a trivial exercise to show that starting from Eq.(17) for a spinor field satisfying Eq.(15) we get the relativistic Hamilton-Jacobi equation.

So, to complete this section we need to determine the rotor $R(\Pi)$. Note that we must have

$$\Pi + eA = mV = mR\gamma^0 R^{-1} \tag{19}$$

and then as shown in Appendix C we find

$$R(\Pi) = \frac{m + (\Pi + eA)\gamma^0}{\left[2\left(m + \Pi_0 + eA_0\right)\right]^{1/2}} \tag{20}$$

4 A Classical Non Linear Dirac-Hestenes Equation

Note that supposing the validity of the classical HJE if instead of taking ψ as in Eq.(14) we write

$$\psi = \varrho^{1/2} R(\Pi) e^{S\gamma^{21}} = \psi_0 e^{S\gamma^{21}} \tag{21}$$

then it is

$$\boldsymbol{\partial}\psi\gamma^{21} = (\boldsymbol{\partial} \ln \psi_0)\psi\gamma^{21} - \boldsymbol{\partial} S\psi \tag{22}$$

and substituting this result in Eq.(12) we get a *nonlinear* Dirac-Hestenes equation, namely [34]

$$\boldsymbol{\partial}\psi\gamma^{21} = -m\psi\gamma^0 + eA\psi - (\boldsymbol{\partial} \ln \psi_0)\psi\gamma^{21}. \tag{23}$$

Remark 2 *Take notice that in this case* $\boldsymbol{\partial} \ln \psi_0 = \frac{1}{2}\boldsymbol{\partial} \ln \rho$.

5 The GHJE which Follows from the General Dirac Equation

In this section we start from the Dirac-Hestenes equation satisfied by a general Dirac-Hestenes spinor field whose representative in the Clifford bundle in a given spin frame is written as

$$\boldsymbol{\psi} = \boldsymbol{\rho}^{1/2} R(\Pi) e^{\frac{\beta\gamma^5}{2}} e^{S\gamma^{21}} = \boldsymbol{\psi}_0 e^{\frac{\beta\gamma^5}{2}} e^{S\gamma^{21}} = e^{\frac{\beta\gamma^5}{2}} \boldsymbol{\psi}_0 e^{S\gamma^{21}}. \tag{24}$$

Thus we have

$$\boldsymbol{\partial}\boldsymbol{\psi}\gamma^{21} = (\boldsymbol{\partial} \ln \boldsymbol{\psi}_0)\boldsymbol{\psi}\gamma^{21} - \boldsymbol{\partial} S\boldsymbol{\psi} - \frac{1}{2}\gamma^5\boldsymbol{\partial}(\ln\beta)\boldsymbol{\psi} \tag{25}$$

and the Dirac-Hestenes becomes

$$(\boldsymbol{\partial} \ln \boldsymbol{\psi}_0)\boldsymbol{\psi}\gamma^{21} - \boldsymbol{\partial} S\boldsymbol{\psi} - \frac{1}{2}\gamma^5\boldsymbol{\partial}(\ln\beta)\boldsymbol{\psi} - m\boldsymbol{\psi}\gamma^0 + eA\boldsymbol{\psi} = 0. \tag{26}$$

equation gives multiplying Eq.(25) on the right by $\boldsymbol{\psi}^{-1}$ and identifying $\Pi = -\boldsymbol{\partial}S$ we get putting

$$\boldsymbol{\psi}\gamma^0\boldsymbol{\psi}^{-1} = e^{\beta\gamma^5}V$$

$$\Pi = me^{\beta\gamma^5}V + eA + (\boldsymbol{\partial}\ln\boldsymbol{\psi}_0)\boldsymbol{\psi}\gamma^{21}\boldsymbol{\psi}^{-1} + \frac{1}{2}\gamma^5\boldsymbol{\partial}(\ln\beta)\boldsymbol{\psi} \tag{27}$$

which can be written as

$$\Pi = m\cos\beta V + eA + m\sin\beta\gamma^5 V + (\boldsymbol{\partial}\ln\boldsymbol{\psi}_0)\boldsymbol{\psi}\gamma^{21}\boldsymbol{\psi}^{-1} + \frac{1}{2}\gamma^5\boldsymbol{\partial}(\ln\beta)\boldsymbol{\psi}. \tag{28}$$

Taking into account that $\Pi = -\boldsymbol{\partial}S$ is a 1-form field we must necessarily have for consistence the following constraint for any solution that implies a genuine classical like equation of motion:

$$\langle e\sin\beta\gamma^5 V + (\boldsymbol{\partial}\ln\boldsymbol{\psi}_0)\boldsymbol{\psi}\gamma^{21}\boldsymbol{\psi}^{-1} + \frac{1}{2}\gamma^5\boldsymbol{\partial}(\ln\beta)\boldsymbol{\psi}\rangle_3 = 0 \tag{29}$$

If constraint given by Eq.(28) is satisfied then the classical like equation of motion for the particle is the following *generalized* Hamilton-Jacobi equation

$$-\boldsymbol{\partial}S = m\cos\beta V + eA + \langle m\sin\beta\gamma^5 V + (\boldsymbol{\partial}\ln\boldsymbol{\psi}_0)\boldsymbol{\psi}\gamma^{21}\boldsymbol{\psi}^{-1} + \frac{1}{2}\gamma^5\boldsymbol{\partial}(\ln\beta)\boldsymbol{\psi}\rangle_1 \tag{30}$$

Remark 3 *Thus the true "quantum potential" is*

$$Q = \langle m\sin\beta\gamma^5 V + (\boldsymbol{\partial}\ln\boldsymbol{\psi}_0)\boldsymbol{\psi}\gamma^{21}\boldsymbol{\psi}^{-1} + \frac{1}{2}\gamma^5\boldsymbol{\partial}(\ln\beta)\boldsymbol{\psi}\rangle_1 \tag{31}$$

which differs considerably form the usual Bohm quantum potential. Moreover and contrary to the usual presentations of the de Broglie-Bohm theory the mass parameter of the particle in the generalized Hamilton-Jacobi equation (Eq.(30)) *is not a constant, instead it is*

$$m' = m\cos\beta. \tag{32}$$

Some results analogous to the ones above but involving classical like equations of motion instead of the generalized Hamilton Jacobi equation (Eq.(30)) has been obtained by Hestenes in memaorble papers [**?**, 17, 18].

6 Description of Spin

In the past sections we associate to a massive and charged particle a Dirac-Hestenes spinor field satisfying Dirac equation which has been shown to be equivalent to the relativistic HJE. We next show [33, 34] how to describe with the same classical spinor field the intrinsic spin of the particle. In order to do that it is necessary to have in mind the concepts of Fermi derivative and the Frenet formalism. For the reader's convenience these concepts are briefly recalled in the Appendix C.

As in previous sections the arena for the motion of particles is Minkowski spacetime $(M \simeq \mathbb{R}^4, \boldsymbol{\eta}, D, \tau_{\boldsymbol{\eta}}, \uparrow)$ for which there are global tetrad frames. So, let $\{\boldsymbol{e}_{\mathbf{a}}\} \in \sec \mathbf{P}_{\mathrm{SO}^e_{1,3}}(M)$ be one of these global tetrad frames. Let $\{\gamma^{\mathbf{a}}\}, \gamma^{\mathbf{a}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ be the dual frame of $\{\boldsymbol{e}_{\mathbf{a}}\}$. Also, let $\{\gamma_{\mathbf{a}}\}$ be the reciprocal frame of $\{\gamma^{\mathbf{a}}\}$, i.e., $\gamma^{\mathbf{a}} \cdot \gamma_{\mathbf{b}} = \delta^{\mathbf{a}}_{\mathbf{b}}$. Suppose moreover that the *reference frame*[7] defined by $\boldsymbol{e}_{\mathbf{0}}$ is in free fall, i.e., $D_{\boldsymbol{e}_{\mathbf{0}}}\boldsymbol{e}_{\mathbf{0}} = 0$ and that the spatial axis along each one of the integral lines of $\boldsymbol{e}_{\mathbf{0}}$ have been constructed by Fermi transport of spinning gyroscopes. This is translated by the requirement that $D_{\boldsymbol{e}_{\mathbf{0}}}\boldsymbol{e}_{\mathbf{i}} = 0$, $\mathbf{i} = 1, 2, 3$, and we have, equivalently $D_{\boldsymbol{e}_0}\gamma^{\mathbf{a}} = 0$. We introduce a spin coframe[8] $\Xi \in P_{\mathrm{Spin}^e_{1,3}}(M)$ such that $s(\pm\Xi) = \{\gamma^{\mathbf{a}}\}$. Now, let Ψ be the representative of an invertible Dirac-Hestenes spinor field over σ (the world line of a spinning particle) in the spin coframe Ξ. Let moreover $\{f_{\mathbf{a}}\}$ be a Frenet coframe over σ such that $f_{\mathbf{0}}$ satisfies $\mathtt{g}(f_{\mathbf{0}}, \ ) = \sigma_*$. Then, since the general form of a representative of an invertible Dirac-Hestenes over σ is $\Psi = \rho^{\frac{1}{2}} e^{\frac{\beta\gamma^5}{2}} \boldsymbol{R}$ we can write for $\beta = 0, \pi$

$$f_{\mathbf{a}} = \Psi \gamma_{\mathbf{a}} \Psi^{-1}. \tag{33}$$

and recalling Eq.(90) from Appendix D we get using Eq.(33) that

$$D_{\boldsymbol{e}_{\mathbf{0}}}\boldsymbol{R} = \frac{1}{2}\Omega_D \boldsymbol{R}, \tag{34}$$

which may be called the *spinor equation of motion* of a classical spinning particle.

Now, let us show that the spinor equation of motion for a *free* particle is equivalent to the classical Dirac-Hestenes equation.

We observe that in this case,of course $\Omega_D = \Omega_{\mathbf{S}}$. Moreover, we can trivially redefine the Frenet frame in such a way as to have $\kappa_3 = 0$. Indeed, this can be done by rotating the original frame with $U = e^{f_{\mathbf{3}} f_{\mathbf{1}} \frac{\alpha}{2}}$ and choosing $\alpha = \arctan\left(-\frac{\kappa_3}{\kappa_1}\right)$. So, in what follows we suppose that this choice has already been done. We are interested in the case where κ_2 is a real constant. Then, Eq.(34) becomes

$$\boldsymbol{D}_{\boldsymbol{e}_0}\boldsymbol{R} = \frac{1}{2}\kappa_{\mathbf{2}} f^{\mathbf{2}} f^{\mathbf{1}} \boldsymbol{R}. \tag{35}$$

The solution of Eq.(35) is

$$\boldsymbol{R} = \boldsymbol{R}_0 \exp\left(\frac{\kappa_2}{2}\gamma^{\mathbf{2}}\gamma^{\mathbf{1}} t\right), \tag{36}$$

where $\boldsymbol{R}_0$ is a constant rotor.

To continue we observe that without any loss of generality we can choose a global tetrad field such that $\gamma^{\mathbf{a}} = \delta^a_\mu dx^\mu$ (where $\{x^\mu\}$ are coordinates in Einstein-Lorentz-Poincaré gauge).

[7] In Relativity theory a general reference frame in a general Lorentzian spacetime $(M \simeq \mathbb{R}^4, \boldsymbol{g}, \boldsymbol{D}, \tau_{\boldsymbol{g}}, \uparrow)$ is a time vector field $\boldsymbol{Z}$ pointing to the future and such that $\boldsymbol{g}(\boldsymbol{Z}, \boldsymbol{Z}) = 1$. In [34] the reader may find a classification of reference frames in Lorentzian spacetimes. For a preliminary classification of reference frames in Riemann-Cartan spacetimes see [14].

[8] Details in [34]

This choice being made we suppose next the existence of a covector field $V \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathrm{g})$ and a classical Dirac-Hestenes spinor field with representative $\psi \in \sec \mathcal{C}\ell(M, \eta)$ in the spin coframe Ξ such that $V|_\sigma = v$ with

$$V|_\sigma = \left. \psi\gamma^0\psi^{-1}\right|_\sigma = \boldsymbol{R}\gamma^0\boldsymbol{R}^{-1}. \tag{37}$$

Then, under all these conditions writing

$$\psi = \psi_0(p)e^{\gamma^{\mathbf{21}}px} \tag{38}$$

where

$$p := \frac{\kappa_2}{2}v \tag{39}$$

and $x = x^\mu\gamma_\mu$ we can rewrite Eq.(35), identifying $\psi_0(p)|_\sigma = \boldsymbol{R}_0$ as:

$$\boldsymbol{D}_{\boldsymbol{e_0}}\boldsymbol{R} = \gamma^0 \cdot \boldsymbol{\partial}\psi = \frac{1}{2}\kappa_2 v^0\psi\gamma^{21}. \tag{40}$$

which reduces to Eq.(35) in a reference frame $\boldsymbol{e}_0$ such that $\boldsymbol{e}_0|_\sigma = v$

Putting $m = -\frac{\kappa_2}{2}$ we immediately get that

$$\boldsymbol{\partial}\psi\gamma^2\gamma^1 - m\psi\gamma^0 = 0. \tag{41}$$

the classical Dirac-Hestenes equation. Note that the signal of κ_2 merely defines the sense of rotation in the $\boldsymbol{e}_2 \wedge \boldsymbol{e}_1$ plane.

The bilinear invariant $\Omega_{\mathcal{S}} \in \sec \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(M, \mathrm{g})$

$$\Omega_{\mathcal{S}} = k\psi\gamma^2\gamma^1\tilde{\psi} \tag{42}$$

the s*pin biform* . Note that for our example $\mathcal{S} = \star\Omega_{\mathcal{S}} \llcorner v = k\psi\gamma^3\tilde{\psi} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \eta)$ and may be called the spin covector.

The classical spinor equation for an electrical spinning particle interacting with an external electromagnetic field can be easily find using the principle of minimal coupling. We find that p must be substituted by the canonical momentum and we arrive at Eq.(17)

Remark 4 *The results just obtained shows that a natural interpretation suggests itself for the plane 'wave function' ψ in the theory just presented. It describes a kind of "probability" field in the sense that it describe a whole set of possible particle trajectories which are non determined, as it is the case n Hamilton-Jacobi theory unless appropriate initial conditions for position and momentum are given* (*something we know is prohibit by Heisenberg uncertain principle*)*.Thus, it is not a physical field in any sense. This last observation agrees with de Broglie opinion in* [7] *and he uses this fact to develop his theory of the double solution. We will now show how de Broglie idea can be realized in our theory.*

7 Realization of De Broglie's Idea of the Double Solution

In this section we will examine only the case of a free particle. We recall that de Broglie [7] tried hard to constructed a theory where the $\psi = R(p)e^{S\gamma^{21}}$ wave satisfying the Dirac-Hestenes equation[9] and describing a free particle has only a statistical significance but that somehow the Dirac-Hestenes equation possesses also a "singular" solution of the form

$$\digamma(x) = \digamma_0(x)e^{S(x)\gamma^{21}} \tag{43}$$

where $\digamma$ is the representative in the Clifford Bundle of a real (not fictitious) *classical* Dirac-Hestenes spinor field describing the motion of a singularity. De Broglie thought that $\digamma_0(x)$ must solve a nonlinear equation. However, this is not the case. We show now that if $\digamma$ satisfies the Dirac-Hestenes equation then $\digamma_0(x)$ satisfies the massless Dirac-Hestenes equation, i.e.,

$$\boldsymbol{\partial}\digamma_0 = 0. \tag{44}$$

Indeed, let us calculate $\boldsymbol{\partial}\digamma(x)$. We have

$$\boldsymbol{\partial}\digamma = (\boldsymbol{\partial}\digamma_0)e^{S\gamma^{21}} + \boldsymbol{\partial}S\digamma_0 e^{S\gamma^{21}}\gamma^{21} \tag{45}$$

But

$$-\boldsymbol{\partial}S = P = mV = mF_0\gamma^0F_0^{-1}. \tag{46}$$

Using this result in Eq.(45)

$$\boldsymbol{\partial}\digamma = (\boldsymbol{\partial}\digamma_0)e^{S\gamma^{21}} - mV\digamma_0 e^{S\gamma^{21}}\gamma^{21} \tag{47}$$

$$\boldsymbol{\partial}\digamma\gamma^{21} = (\boldsymbol{\partial}\digamma_0)e^{S\gamma^{21}}\gamma^{21} + m\digamma\gamma^{o} \tag{48}$$

and since $\digamma$ satisfies the Dirac-Hestenes equation we have that $\boldsymbol{\partial}\digamma_0 = 0$

The question that immediately comes to mind is:

May Eq.(44) *possess solutions satisfying the constraint* (**??**) *describing the motion of a massive* (*and spinning*) *free particle moving with a subluminal velocity* v*?*

The answer to that question is *yes*. Indeed, introducing the potential $\mathcal{A} \in \sec\bigwedge^1 T^*M \hookrightarrow \sec\mathcal{C}\ell(M, \mathtt{g})$ and defining

$$\digamma_0 := \boldsymbol{\partial}\mathcal{A} \tag{49}$$

we have immediately from Eq.(44) that

$$\boldsymbol{\partial}^2\mathcal{A} = 0. \tag{50}$$

[9] Of course, de Broglie used the standard matrix formulation for the Dirac equation since the idea of Dirac-Hestenes spinor fields where not known when he was investigating his theory the double solution.

Now, we found long ago [29, 30] that Eq.(50) has *subluminal soliton like* solutions. Putting $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$ and supposing for simplicity that the wave is moving in the z-direction a subluminall solution rigidly moving with velocity (1-form) is

$$\mathcal{A} = \mathcal{C}\frac{\sin m\xi}{\xi}e^{(\omega t - kz)}\gamma^1. \tag{51}$$

with

$$\begin{gathered}
\xi = [(x)^2 + (y)^2 + \Gamma^2(z - vt)^2]^{\frac{1}{2}}, \\
\Gamma = (1 - \mathtt{v}^2)^{-\frac{1}{2}}, \\
\omega^2 - k^2 = m^2, \\
0 < \mathtt{v} = \frac{d\omega}{dk} < 1.
\end{gathered}$$

and where $\mathcal{C}$ is a constant. Then, the moving soliton like object realizing de Broglie dream is

$$F_0 = \mathcal{C}\boldsymbol{\partial}\left(\frac{\sin m\xi}{\xi}e^{(\omega t - kz)}\right)\gamma^1. \tag{52}$$

Observe that in the rest frame of the soliton $F_0 \in \sec\bigwedge^2 T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$ is simply

$$F_0 = \mathcal{C}m\left(\frac{\sin m[(x)^2 + (y)^2 + (z)^2]^{\frac{1}{2}}}{[(x)^2 + (y)^2 + (z)^2]^{\frac{1}{2}}}e^{mt'}\right)\gamma'^{01} \tag{53}$$

Remark 5 *Since* $\gamma'^{01} = u\gamma^{01}u^{-1}$ *and* $u = e^{\chi\gamma^{30}}$ *we see that* $F_0\gamma^0F_0^{-1} \in \sec\bigwedge^1 T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$

and thus qualify qualifies $F_0\gamma^0F_0^{-1}$ *as a* 1*-form velocity field according to* Eq.(46).

Some nontrivial problems that need to be investigate are: How the field F_0 behaves when it meets an obstacle, e.g., a double slit apparatus? How to describe the motion of more than one F like fields moving in Minkowski spacetime? We will return to an investigation of this problem in another paper.

8 Conclusions

We proved that the relativistic Hamilton-Jacobi equation for a massive particle charged particle (moving in Minkowski spacetime) interacting with an electromagnetic field is equivalent to a Dirac-Hestenes equation[10] satisfied by a special

[10] Which is the representative of the usual Dirac equation in the Clifford bundle (a result recalled in Appendix A).

class of Dirac spinor fields[11] (characterized for having the Takabayashi angle function equal to 0 or π). Also, any Dirac-Hestenes equation satisfied by these classical spinor fields implies in a corresponding relativistic Hamilton-Jacobi equation.

After proving these result we show that since general spinor fields which are solutions of the Dirac equation in an external potential have in general Takabayashi angle functions [1, 6] which are not constant functions) the corresponding derived generalized Hamilton-Jacobi equation (GHJE) besides having a quantum potential which must satisfy a very severe constraint (see Eq.(29)) has also a variable mass (which is a function of the Takabayashi angle function). So, the usual equations derived from Schrödinger equation used in simulations of, e.g., the double slit experiments do not take into account that the mass of the particle which comes from the generalized HJE becomes a variable. This eventually must explain the discrepancies found between theoretical predictions from the de Broglie and Bohm formalism and some results of experiments and founded inconsistencies (such as necessity of surreal trajectories) as related, by several authors (cited in the Section 1)[12]. We have also shown that de Broglie's double solution theory can be realized at least for the case of a spinning free particle. However contrary to de Broglie's suggestion [7] we found that the physical field $\digamma_0$ in Eq.(44) satisfies the massless Dirac equation instead of a nonlinear field This finding suggested the investigation of many questions, e.g.:

(i) how the field $\digamma_0$ behaves when it meets an obstacle, e.g., a double slit apparatus?

(ii) how to describe the motion of more than one $\digamma$ like fields moving in Minkowski spacetime?

These issues will be investigated in another paper.

All the above results have been proved using the powerful Clifford and spin-Clifford bundles formalism where Dirac spinor fields are represented by an equivalence class of even sections of the Clifford bundle $\mathcal{C}\ell(M,\eta)$ of differential forms. The Appendices present this formalism, the nomenclature and proofs of some results appearing in the main text.

Finally, while preparing this version of the paper we have been informed by Professor Basil Hiley of his papers [19, 20, 21]. In particular, the last two papers contains material related to our paper but with different and conflicting results which we intend to discuss in another publication.

A Preliminaries

Let M be a four dimensional, real, connected, paracompact and non-compact manifold. We recall that a Lorentzian manifold as a pair $(M,\boldsymbol{g})$, where $\boldsymbol{g} \in \sec T_2^0 M$ is a Lorentzian metric of signature $(1,3)$, i.e., $\forall x \in M, T_xM \simeq T_x^*M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the Minkowski vector space. We define a Lorentzian spacetime M as pentuple $(M,\boldsymbol{g},\boldsymbol{D},\tau_{\boldsymbol{g}},\uparrow)$, where $(M,\boldsymbol{g},\tau_{\boldsymbol{g}},\uparrow))$ is an oriented Lorentzian

[11] These special class of spinor fields will be called classical spinor fields.

[12] But on this issue see also [19, 8].

manifold (oriented by $\tau_{\boldsymbol{g}}$) and time oriented by $\uparrow$, and $\boldsymbol{D}$ is the Levi-Civita connection of $\boldsymbol{g}$. Let $\mathcal{U} \subseteq M$ be an open set covered by coordinates $\{x^\mu\}$. Let $\{e_\mu = \partial_\mu\}$ be a coordinate basis of $T\mathcal{U}$ and $\{\boldsymbol{\vartheta}^\mu = dx^\mu\}$ the dual basis on $T^*\mathcal{U}$, i.e., $\boldsymbol{\vartheta}^\mu(\partial_\nu) = \delta^\mu_\nu$. If $\boldsymbol{g} = g_{\mu\nu}\boldsymbol{\vartheta}^\mu \otimes \boldsymbol{\vartheta}^\nu$ is the metric on $T\mathcal{U}$ we denote by $\mathtt{g} = g^{\mu\nu}\boldsymbol{\partial}_\mu \otimes \boldsymbol{\partial}_\nu$ the metric of $T^*\mathcal{U}$, such that $g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu$. We introduce also $\{\boldsymbol{\partial}^\mu\}$ and $\{\boldsymbol{\vartheta}_\mu\}$, respectively, as the reciprocal bases of $\{e_\mu\}$ and $\{\boldsymbol{\vartheta}_\mu\}$, i.e., we have

$$\boldsymbol{g}(\boldsymbol{\partial}_\nu, \boldsymbol{\partial}^\mu) = \delta^\mu_\nu, \quad \mathtt{g}(\boldsymbol{\vartheta}^\mu, \boldsymbol{\vartheta}_\nu) = \delta^\mu_\nu. \tag{54}$$

In what follows $\mathbf{P}_{\mathrm{SO}^e_{1,3}}(M, \boldsymbol{g})$ ($P_{\mathrm{SO}^e_{1,3}}(M, \mathtt{g})$) denotes the principal bundle of oriented Lorentz tetrads (cotetrads).

A *spin structure* for a general m-dimensional manifold M (with $m = p + q$) equipped with a metric field $\boldsymbol{g}$ is a principal fiber bundle $\pi_s : P_{\mathrm{Spin}^e_{p,q}}(M, \mathtt{g}) \to M$, (called the Spin Frame Bundle) with group $\mathrm{Spin}^e_{p,q}$ such that there exists a map

$$\Lambda : P_{\mathrm{Spin}^e_{p,q}}(M, \mathtt{g}) \to P_{\mathrm{SO}^e_{p,q}}(M, \mathtt{g}), \tag{55}$$

satisfying the following conditions:

Definition 6 **(i)** $\pi(\Lambda(p)) = \pi_s(p), \forall p \in P_{\mathrm{Spin}^e_{p,q}}(M, \mathtt{g})$, *where* π *is the projection map of the bundle* $\pi : P_{\mathrm{SO}^e_{p,q}}(M, \mathtt{g}) \to M$.

(ii) $\Lambda(pu) = \Lambda(p)\mathrm{Ad}_u, \forall p \in P_{\mathrm{Spin}^e_{p,q}}(M, \mathtt{g})$ *and* $\mathrm{Ad} : \mathrm{Spin}^e_{p,q} \to \mathrm{SO}^e_{p,q}$, $\mathrm{Ad}_u(a) = uau^{-1}$.

Any section of $P_{\mathrm{Spin}^e_{p,q}}(M, \mathtt{g})$ is called a *spin frame field (* or simply a spin frame). We shall use the symbol $\Xi \in \sec P_{\mathrm{Spin}^e_{p,q}}(M, \mathtt{g})$ to denoted a spin frame.

It can be shown that[13] [34]:

$$\mathcal{C}\ell(M, \mathtt{g}) = P_{\mathrm{SO}^e_{1,3}}(M, \mathtt{g}) \times_\rho \mathbb{R}_{1,3} = P_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g}) \times_{Ad} \mathbb{R}_{1,3}, \tag{56}$$

and since[14] $\bigwedge TM \hookrightarrow \mathcal{C}\ell(M, \mathtt{g})$, sections of $\mathcal{C}\ell(M, \mathtt{g})$ (the Clifford fields) can be represented as a sum of non homogeneous differential forms.

For any parallelizable spacetime structure (as it is the case of Minkowski spacetime used in the main text), we introduce the global tetrad basis $\boldsymbol{e}_\alpha, \alpha = 0, 1, 2, 3$ on TM and in T^*M the cotetrad basis on $\{\boldsymbol{\gamma}^\alpha\}$, which are dual basis. We introduce the reciprocal basis $\{\boldsymbol{e}^\alpha\}$ and $\{\boldsymbol{\gamma}_\alpha\}$ of $\{\boldsymbol{e}_\alpha\}$ and $\{\boldsymbol{\gamma}^\alpha\}$ satisfying

$$\boldsymbol{g}(\boldsymbol{e}_\alpha, \boldsymbol{e}^\beta) = \delta^\beta_\alpha, \quad \mathtt{g}(\boldsymbol{\gamma}^\beta, \boldsymbol{\gamma}_\alpha) = \delta^\beta_\alpha. \tag{57}$$

Moreover, recall that[15]

$$\boldsymbol{g} = \eta_{\alpha\beta}\boldsymbol{\gamma}^\alpha \otimes \boldsymbol{\gamma}^\beta = \eta^{\alpha\beta}\boldsymbol{\gamma}_\alpha \otimes \boldsymbol{\gamma}_\beta, \quad \mathtt{g} = \eta^{\alpha\beta}\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta = \eta_{\alpha\beta}\boldsymbol{e}^\alpha \otimes \boldsymbol{e}^\beta. \tag{58}$$

[13] Where $Ad : \mathrm{Spin}^e_{1,3} \to \mathrm{End}(\mathbb{R}_{1,3})$ is such that $Ad(u)a = uau^{-1}$. And $\rho : \mathrm{SO}^e_{1,3} \to \mathrm{End}(\mathbb{R}_{1,3})$ is the natural action of $\mathrm{SO}^e_{1,3}$ on $\mathbb{R}_{1,3}$.

[14] Given the objets A and B, $A \hookrightarrow B$ means as usual that A is embedded in B and moreover, $A \subseteq B$. In particular, recall that there is a canonical vector space isomorphism between $\bigwedge \mathbb{R}^{1,3}$ and $\mathbb{R}_{1,3}$, which is written $\bigwedge \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$. Details in [5, 25].

[15] Where the matrix with entries $\eta_{\alpha\beta}$ (or $\eta^{\alpha\beta}$) is the diagonal matrix $(1, -1, -1, -1)$.

In this work we have that exists a spin structure on the 4-dimensional Lorentzian manifold $(M, \boldsymbol{g})$, since M is parallelizable, i.e., $P_{\mathrm{SO}^e_{1,3}}(M, \mathtt{g})$ is trivial, because of the following result due to Geroch [11, 12]:

Theorem 7 *For a 4−dimensional Lorentzian manifold $(M, \boldsymbol{g})$, a spin structure exists if and only if $P_{\mathrm{SO}^e_{1,3}}(M, \mathtt{g})$ is a trivial bundle* [11, 12]

The basis $\boldsymbol{\gamma}^\alpha|_p$ of $T_pM \simeq \mathbb{R}^{1,3}, p \in M$, generates the algebra $\mathcal{C}\ell(T_pM, \mathtt{g}) \simeq \mathbb{R}_{1,3}$. We have that [34]

$$\mathrm{e} = \frac{1}{2}(1 + \boldsymbol{\gamma}^0) \in \mathbb{R}_{1,3}$$

is a primitive idempotent of $\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$ (the so called spacetime algebra)[16] and

$$\mathrm{f} = \frac{1}{2}(1 + \boldsymbol{\gamma}^0)\frac{1}{2}(1 + i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^1) \in \mathbb{C} \otimes \mathbb{R}_{1,3}$$

is a primitive idempotent of $\mathbb{C} \otimes \mathbb{R}_{1,3}$. Now, let $I = \mathbb{R}_{1,3}\mathrm{e}$ and $I_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{R}_{1,3}\mathrm{f}$ be respectively the minimal left ideals of $\mathbb{R}_{1,3}$ and $\mathbb{C} \otimes \mathbb{R}_{1,3}$ generated by e and f. Any $\phi \in I$ can be written as

$$\phi = \psi \mathrm{e}$$

with $\psi \in \mathbb{R}^0_{1,3}$. Analogously, any $\phi \in I_{\mathbb{C}}$ can be written as

$$\psi \mathrm{e}\frac{1}{2}(1 + i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^1)$$

with $\psi \in \mathbb{R}^0_{1,3}$. Recall moreover that $\mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$. We can verify that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is a primitive idempotent of $\mathbb{C}(4)$ which is a matrix representation of f. In that way, there is a bijection between column spinors, i.e., elements of $\mathbb{C}^4$ and the elements of $I_{\mathbb{C}}$.

Recalling that $\mathrm{Spin}^e_{1,3} \hookrightarrow \mathbb{R}^0_{1,3}$, we give:

Definition 8 *The left (respectively right) real spin-Clifford bundle of the spin manifold M is the vector bundle $\mathcal{C}\ell^l_{\mathrm{Spin}}(M, \mathtt{g}) = P_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g}) \times_l \mathbb{R}_{1,3}$ (respectively $\mathcal{C}\ell^r_{\mathrm{Spin}}(M, \mathtt{g}) = P_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g}) \times_r \mathbb{R}_{1,3}$) where l is the representation of $\mathrm{Spin}^e_{1,3}$ on $\mathbb{R}_{1,3}$ given by $l(a)x = ax$ (respectively, where r is the representation of $\mathrm{Spin}^e_{1,3}$ on $\mathbb{R}_{1,3}$ given by $r(a)x = xa^{-1}$). Sections of $\mathcal{C}\ell^l_{\mathrm{Spin}}(M, \mathtt{g})$ are called left spin-Clifford fields (respectively right spin-Clifford fields).*

[16] We recall that $\mathcal{C}\ell(T^*_xM, \eta) \simeq \mathbb{R}_{1,3}$ the so-called spacetime algebra. Also the even subalgebra of $\mathbb{R}_{1,3}$ denoted $\mathbb{R}^0_{1,3}$ is isomorphic to te Pauli algebra $\mathbb{R}_{3,0}$, i.e., $\mathbb{R}^0_{1,3} \simeq \mathbb{R}_{3,0}$. The even subalgebra of the Pauli algebra $\mathbb{R}^0_{3,0} := \mathbb{R}^{00}_{3,0}$ is the quaternion algebra $\mathbb{R}_{0,2}$, i.e., $\mathbb{R}_{0,2} \simeq \mathbb{R}^0_{3,0}$. Moreover we have the identifications: $\mathrm{Spin}^0_{1,3} \simeq \mathrm{Sl}(2, \mathbb{C})$, $\mathrm{Spin}_{3,0} \simeq \mathrm{SU}(2)$. For the Lie algebras of these groups we have $\mathrm{spin}^0_{1,3} \simeq \mathrm{sl}(2, \mathbb{C})$, $\mathrm{su}(2) \simeq \mathrm{spin}_{3,0}$. The important fact to keep in mind for the understanding of some of the identificastions we done below is that $\mathrm{Spin}^0_{1,3}, \mathrm{spin}^0_{1,3} \subset \mathbb{R}_{3,0} \subset \mathbb{R}_{1,3}$ and $\mathrm{Spin}_{3,0}, \mathrm{spin}_{3,0} \subset \mathbb{R}_{0,2} \subset \mathbb{R}^0_{1,3} \subset \mathbb{R}_{1,3}$.

Definition 9 *Let* $\mathbf{e}, \mathbf{f} \in \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$ *be a primitive global idempotents* [17], *respectively* $\mathbf{e^r}, \mathbf{f}^r \in \mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$*, and let* $I(M, \mathtt{g})$ *be the subbundle of* $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$ *generated by the idempotent, that is, if* $\mathbf{\Psi}$ *is a section of* $I(M, \mathtt{g}) \subset \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$*, we have*

$$\mathbf{\Psi e} = \mathbf{\Psi}, \tag{59}$$

A section $\mathbf{\Psi}$ *of* $I(M, \mathtt{g})$ *is called a left ideal algebraic spinor field.*

Definition 10 *A Dirac-Hestenes spinor field* (DHSF) *associated with* $\mathbf{\Psi}$ *is a section*[18] Ψ *of* $\mathcal{C}\ell^{0l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g}) \subset \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$ *such that*[19]

$$\mathbf{\Psi} = \Psi \mathbf{e}. \tag{60}$$

Definition 11 *We denote the complexified left spin-Clifford bundle by*

$$\mathbb{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g}) = P_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g}) \times_l \mathbb{C} \otimes \mathbb{R}_{1,3} \equiv P_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g}) \times_l \mathbb{R}_{1,4}.$$

Definition 12 *An equivalent definition of a* DHSF *is the following. Let* $\mathbf{\Psi} \in \sec \mathbb{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$ *such that*

$$\mathbf{\Psi f} = \mathbf{\Psi}.$$

Then a DHSF *associated with* $\mathbf{\Psi}$ *is an even section* Ψ *of* $\mathcal{C}\ell^{0l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g}) \subset \mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$ *such that*

$$\mathbf{\Psi} = \Psi \mathbf{f}. \tag{61}$$

The matrix representations of Ψ and $\mathbf{\Psi}$ in $\mathbb{C}(4)$ (denoted by the same letter) in the given spin basis are[20]

$$\Psi = \begin{pmatrix} \psi_1 & -\psi_2^* & \psi_3 & \psi_4^* \\ \psi_2 & \psi_1^* & \psi_4 & -\psi_3^* \\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^* \end{pmatrix}, \quad \mathbf{\Psi} = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \tag{62}$$

A.1 The Hidden Geometrical Meaning of Spinors

DHSFs unveil the hidden geometrical meaning of spinors (and spinor fields). Indeed, consider $v \in \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ a timelike *covector* such that $v^2 = 1$. The linear mapping, belonging to $\mathrm{SO}^{e}_{1,3}$

$$v \mapsto RvR^{-1} = Rv\tilde{R} = w, \; R \in \mathrm{Spin}^{e}_{1,3}, \tag{63}$$

[17] We know that global primitive idempotents exist because M is parallelizable.

$$\mathbf{e} = [(\mathbf{\Xi_0}, \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1} + \boldsymbol{\gamma}^{\mathbf{0}}))], \mathbf{f} = [(\mathbf{\Xi}_0, \frac{1}{2}(1 + \boldsymbol{\gamma}^0)\frac{1}{2}(1 + i\boldsymbol{\gamma}^2\boldsymbol{\gamma}^1))]$$

[18] $\mathcal{C}\ell^{0l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$ denotes the even subbundle of $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathtt{g})$

[19] For any $\mathbf{\Psi}$ the DHSF always exist, see [34].

[20] Note that in Eq.(62) the ψ_i are functions from M to $\mathbb{R}$.

define a new covector w such that $w^2 = 1$. We can therefore fix a covector v and obtain all other unit timelike covectors by applying this mapping. This same procedure can be generalized to obtain any type of timelike covector starting from a fixed unit covector v. We define the linear mapping

$$v \mapsto \psi v \tilde{\psi} = z \tag{64}$$

to obtain $z^2 = \rho^2 > 0$. Since z can be written as $z = \rho R v \tilde{R}$, we need

$$\psi v \tilde{\psi} = \rho R v \tilde{R}. \tag{65}$$

If we write $\psi = \rho^{\frac{1}{2}} M R$ we need that $M v \tilde{M} = v$ and the most general solution is $M = e^{\frac{\boldsymbol{\tau_g} \beta}{2}}$, where $\boldsymbol{\tau_g} = \gamma^0\gamma^1\gamma^2\gamma^3 \in \bigwedge^4 \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$ and $\beta \in \mathbb{R}$ is called the Takabayasi angle [34, 37]. Then IT follows that ψ is of the form

$$\psi = \rho^{\frac{1}{2}} e^{\frac{\boldsymbol{\tau_g} \beta}{2}} R. \tag{66}$$

Now, Eq.(66) shows that $\psi \in \mathbb{R}^0_{1,3} \simeq \mathbb{R}_{3,0}$. Moreover, we have that $\psi\tilde{\psi} \neq 0$ since

$$\psi\tilde{\psi} = \rho e^{\boldsymbol{\tau_g}\beta} = (\rho \cos \beta) + \boldsymbol{\tau_g}(\rho \sin \beta). \tag{67}$$

A representative of a DHSF Ψ in the Clifford bundle $\mathcal{C}\ell(M, \mathtt{g})$ relative to a spin frame $\boldsymbol{\Xi}_u$ is a section $\boldsymbol{\psi}_{\boldsymbol{\Xi}_u} = [(\boldsymbol{\Xi}_u, \psi_{\boldsymbol{\Xi}_u})]$ of $\mathcal{C}\ell^0(M, \mathtt{g})$ where $\psi_{\boldsymbol{\Xi}_u} \in \mathbb{R}^0_{1,3} \simeq \mathbb{R}_{3,0}$. So a DHSF such $\boldsymbol{\psi}_{\boldsymbol{\Xi}_u}\tilde{\boldsymbol{\psi}}_{\boldsymbol{\Xi}_u} \neq 0$ induces a linear mapping induced by Eq.(64), which *rotates* a covector field and *dilate* it.

B Description of the Dirac Equation in the Clifford Bundle

In the main text we utilized as arena for the motion of particles (and fields) the Minkowski spacetime structure $(M \simeq \mathbb{R}^4, \boldsymbol{\eta}, D, \tau_\eta)$ $\boldsymbol{\eta} \in \sec T^2_0 M$ is Minkowski metric and D is the Levi-Civita connection of $\boldsymbol{\eta}$. Also, $\tau_\eta \in \sec \bigwedge^4 T^* M$ defines an orientation. We denote by $\eta \in \sec T^0_2 M$ the metric of the cotangent bundle. It is defined as follows. Let $\{x^\mu\}$ be coordinates for M in the Einstein-Lorentz-Poincaré gauge [34]. Let $\{\boldsymbol{e}_\mu = \partial/\partial x^\mu\}$ a basis for TM and $\{\gamma^\mu = dx^\mu\}$ the corresponding dual basis for T^*M, i.e., $\gamma^\mu(\boldsymbol{e}_\alpha) = \delta^\mu_\alpha$. Then, if $\boldsymbol{\eta} = \eta_{\mu\nu}\gamma^\mu \otimes \gamma^\nu$ then $\eta = \eta^{\mu\nu}\boldsymbol{e}_\mu \otimes \boldsymbol{e}_\nu$, where the matrix with entries $\eta_{\mu\nu}$ and the one with entries $\eta^{\mu\nu}$ are the equal to the diagonal matrix $\mathrm{diag}(1, -1, -1, -1)$. If $a, b \in \sec \bigwedge^1 T^*M$ we write $a \cdot b = \eta(a, b)$. We also denote by $\langle \gamma_\mu \rangle$ the reciprocal basis of $\{\gamma^\mu = dx^\mu\}$, which satisfies $\gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu$.

We denote the Clifford bundle of differential forms in Minkowski spacetime by $\mathcal{C}\ell(M, \eta)$ and use notations and conventions in what follows as in [34] and recall the fundamental relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}. \tag{68}$$

If $\{\boldsymbol{\gamma}^{\mu}, \quad \mu = 0,1,2,3\}$ are the Dirac gamma matrices in the *standard representation* and $\{\gamma_{\mu}, \ \mu = 0,1,2,3\}$ are as introduced above, we define

$$\sigma_k := \gamma_k \gamma_0 \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell^0(M,\eta), \ k = 1,2,3, \tag{69}$$

$$\mathbf{i} = \gamma_5 := \gamma_0\gamma_1\gamma_2\gamma_3 \in \sec \bigwedge^4 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta), \tag{70}$$

$$\boldsymbol{\gamma}_5 := \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 \in \mathbb{C}(4) \tag{71}$$

Noting that M is parallelizable, given a global spin frame a *covariant spinor field* can be taken as a mapping $\boldsymbol{\Psi} : M \to \mathbb{C}^4$ In standard representation of the gamma matrices where ($i = \sqrt{-1}$, $\boldsymbol{\phi}, \boldsymbol{\varsigma} : M \to \mathbb{C}^2$) $\boldsymbol{\psi}$ is given by

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\varsigma} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} m^0 + im^3 \\ -m^2 + im^1 \end{pmatrix} \\ \begin{pmatrix} n^0 + in^3 \\ -n^2 + in^1 \end{pmatrix} \end{pmatrix}, \tag{72}$$

and to $\boldsymbol{\Psi}$ there corresponds the **DHSF** $\psi \in \sec \mathcal{C}\ell^0(M,\eta)$ given by[21]

$$\psi = \phi + \varsigma\sigma_3 = (m^0 + m^k \mathbf{i}\sigma_k) + (n^0 + n^k \mathbf{i}\sigma_k)\sigma_3. \tag{73}$$

We then have the useful formulas in Eq.(74) below that one can use to immediately translate results of the standard matrix formalism in the language of the Clifford bundle formalism and vice-versa[22]

$$\begin{aligned} \boldsymbol{\gamma}_{\mu}\boldsymbol{\Psi} &\leftrightarrow \gamma_{\mu}\psi\gamma_0, \\ i\boldsymbol{\Psi} &\leftrightarrow \psi\gamma_{21} = \psi\mathbf{i}\sigma_3, \\ i\boldsymbol{\gamma}_5\boldsymbol{\Psi} &\leftrightarrow \psi\sigma_3 = \psi\gamma_3\gamma_0, \\ \bar{\boldsymbol{\Psi}} = \boldsymbol{\Psi}^{\dagger}\boldsymbol{\gamma}^0 &\leftrightarrow \tilde{\psi}, \\ \boldsymbol{\Psi}^{\dagger} &\leftrightarrow \gamma_0\tilde{\psi}\gamma_0, \\ \boldsymbol{\Psi}^{*} &\leftrightarrow -\gamma_2\psi\gamma_2. \end{aligned} \tag{74}$$

Using the above dictionary the standard (free) Dirac equation[23] for a Dirac spinor field $\boldsymbol{\psi} : M \to \mathbb{C}^4$

$$i\boldsymbol{\gamma}^{\mu}\partial_{\mu}\boldsymbol{\Psi} - m\boldsymbol{\Psi} = 0 \tag{75}$$

translates immediately in the so-called Dirac-Hestenes equation, i.e.,

$$\boldsymbol{\partial}\psi\gamma_{21} - m\psi\gamma_0 = 0. \tag{76}$$

[21] Remember the identification:

$$\mathbb{C}(4) \simeq \mathbb{R}_{4,1} \supseteq \mathbb{R}^0_{4,1} \simeq \mathbb{R}_{1,3}.$$

[22] $\tilde{\psi}$ is the reverse of ψ. If $A_r \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\eta)$ then $\tilde{A}_r = (-1)^{\frac{r}{2}(r-1)} A_r$.

[23] $\partial_{\mu} := \frac{\partial}{\partial x^{\mu}}$.

C Proof of Eq.(20)

From Eqs. (15), (16) and (17) we have that

$$\Pi + eA = mV = mR\gamma^0 R^{-1} = \boldsymbol{L}\gamma^0 \boldsymbol{L}^{-1} \tag{77}$$

This means that the rotor R $R : M \ni x \mapsto R(x) \in \mathrm{Spin}_{1,3}^0$ must be a boost. We also know that every boost must be the exponential of a biform [38, 26].

Now, we observe that for any $x \in M$ if $X(x)$ is such that $X(x) \lrcorner ((\Pi + eA) \wedge \gamma^0) = 0$ we will have

$$RXR^{-1} = X \tag{78}$$

and thus we conclude that R must be a biform proportional to $C = \gamma^0 \wedge (\Pi + eA)$. Indeed, since $(\Pi + eA) \wedge \gamma^0$ anticommutes with $(\Pi + eA)$ and γ^0 we have that

$$\frac{1}{m}\Pi + eA = R\gamma^0 R^{-1} = R^2\gamma^0 = V \tag{79}$$

As we can verify by direct computation the have

$$R(V) = \frac{1 + V\gamma^0}{1\left[2\left(1 + V_0\right)\right]^{1/2}} \tag{80}$$

and thus we can write

$$R(\Pi) = \frac{m + (\Pi + eA)\gamma^0}{\left[2\left(m + \Pi_0 + A_0\right)\right]^{1/2}} \tag{81}$$

with

$$\cosh\chi = \frac{1}{m}(\Pi_0 + A_0) \tag{82}$$

Now, without any difficult we can show that

$$R(\Pi) = e^{\left(\frac{\chi}{2}\frac{(\Pi + eA)\wedge\gamma^0}{|(\Pi + eA)\wedge\gamma^0|}\right)} \tag{83}$$

For the case where $A = 0$ the canonical momentum is P, with $P^2 = m^2$. In this case [9] Eq.(81) is

$$R(P) = \frac{m + V\gamma^0}{\left[2m\left(m + P^0\right)\right]^{1/2}}. \tag{84}$$

D Rotation and Fermi Transport in the Clifford Bundle Formalism

Let $\sigma : \mathbb{R} \supset I \to M$, $\tau \mapsto \sigma(\tau)$ be a timelike curve on Minkowski spacetime. Let $\mathbf{Y}$ be a vector field over σ. As well known [35] in order for an observer (following the curve σ to decide when a unitary vector $\mathbf{Y} \in (\sigma_{*\tau})^{\perp}$ has the same spatial direction of the unitary vector $\mathbf{Y}' \in (\sigma_{*\tau'})^{\perp}$ $(\tau' \neq \tau)$, he has to introduce the concept of the *Fermi-Walker connection*. We have the

Proposition 13 *There exists one and only one connection $\mathcal{F}$ over σ, such that*

$$\mathcal{F}_{\mathbf{X}}\mathbf{Y} = [\mathbf{p}(\sigma^{*}\boldsymbol{D})_{\mathbf{X}}\mathbf{p} + \mathbf{q}(\sigma^{*}\boldsymbol{D})_{\mathbf{X}}\mathbf{q}]\mathbf{Y} \tag{85}$$

for all vector fields $\mathbf{X}$ on I and for all vector fields $\mathbf{Y}$ over σ.

In Eq.(85) $\sigma^{*}\boldsymbol{D}$ is the *induced* connection over σ of the Levi-Civita connection $\boldsymbol{D}$ and $\mathcal{F}$ is called the Fermi-Walker connection over σ, and we shall use the notations $\mathcal{F}_{\sigma_*}, \mathcal{F}/d\tau$ or $\mathcal{F}_{\boldsymbol{\varepsilon}_0}$ (see below) when convenient. We also will write (by abuse of language) only $\boldsymbol{D}$, as usual, for $\sigma^{*}\boldsymbol{D}$ in what follows. A proof of the theorem can be found, e.g., in [34]

We recall that a *moving (orthonormal) frame* $\{\epsilon_{\mathbf{a}}\}$ over σ is an orthonormal basis for $T_{\sigma(I)}M$ with $\epsilon_{\mathbf{0}} = \sigma_*$. The set $\{\varepsilon^{\mathbf{a}}\}$, $\varepsilon^{\mathbf{a}} \in \sec T^{*}_{\sigma(I)}M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g})$ is the dual comoving frame on σ, i.e., $\varepsilon^{\mathbf{a}}(\epsilon_{\mathbf{b}}) = \delta^{\mathbf{a}}_{\mathbf{b}}$.The set $\{\varepsilon_{\mathbf{a}}\}$, $\varepsilon_{\mathbf{a}} \in \sec T^{*}_{\sigma(I)}M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g})$, with $\varepsilon^{\mathbf{a}} \cdot \varepsilon_{\mathbf{b}} = \delta^{\mathbf{a}}_{\mathbf{b}}$ is the reciprocal frame of $\{\varepsilon^{\mathbf{a}}\}$.

Let $\mathbf{X}$ and $\mathbf{Y}$ be vector fields over σ and $Y = \boldsymbol{g}(\mathbf{X},\ ), Y = \boldsymbol{g}(\mathbf{Y},\ ) \in \sec T^{*}_{\sigma(I)}M \hookrightarrow \mathcal{C}\ell(M,\mathtt{g})$ the physically equivalent 1-form fields. We have the

Proposition 14 *Let be X, Y be form fields over σ, as defined above. The Fermi-Walker connection $\mathcal{F}$ satisfies the properties*

(a) $\quad \mathcal{F}_{\epsilon_o}Y = \boldsymbol{D}_{\epsilon_{\mathbf{0}}}Y - (\varepsilon_{\mathbf{0}} \cdot Y)a + (a \cdot Y)\varepsilon_0$,
where $a = \boldsymbol{D}_{\epsilon_{\mathbf{0}}}\varepsilon_0$, *is the* (1-*form*) *acceleration.*

(b) $\quad \frac{d}{d\tau}\ (X \cdot Y) = \mathcal{F}_{\epsilon_{\mathbf{0}}}X \cdot Y + X \cdot \mathcal{F}_{\epsilon_{\mathbf{0}}}Y$

(c) $\quad \mathcal{F}_{\epsilon_{\mathbf{0}}}\varepsilon_{\mathbf{0}} = 0$

(d) $\quad$ *If $X, \mathbf{Y}$ are vector fields on σ such that $X_u, \mathbf{Y}_u \in H_u$ $\forall u \in I$ then*[24] $\mathtt{g}(\mathcal{F}_{\epsilon_{\mathbf{0}}}X,)|_u$ *and* $g(\mathcal{F}_{\epsilon_{\mathbf{0}}}Y,)|_u \in H_u$, $\forall u$ *and*

$$\mathcal{F}_{\epsilon_{\mathbf{0}}}X \cdot Y = \boldsymbol{D}_{\epsilon_{\mathbf{0}}}X \cdot Y \tag{86}$$

For a proof, see, e.g., [34]

Now, let $Y_0 \in \sec T^{*}_{\sigma\tau_0}M$. Then, by a well known property of connections there exists one and only one 1-form field Y over σ such that $\mathcal{F}_{\epsilon_o}Y = 0$ and $Y(\tau_0) = Y_0$. So, if $\{\varepsilon_{\mathbf{a}}|_{\tau_0}\}$ is an orthonormal basis for $T^{*}_{\sigma\tau_0}M$ ($\varepsilon_{\mathbf{a}}|_{\tau_0} \in T^{*}_{\sigma\tau_0}M$, $\mathbf{a} = 0,1,2,3$) we have that the $\varepsilon_{\mathbf{a}}$'s such that $\mathcal{F}_{\epsilon_{\mathbf{o}}}\varepsilon_{\mathbf{a}} = 0$ are orthonormal for any τ, as follows from (b) in proposition 14.

We then, agree to say that $Y_1 \in H^{*}_{\tau_1}$ and $Y_2 \in H^{*}_{\tau_2}$ have the same spatial direction if and only if $Y_1 = a^i\ \varepsilon_i|_{\tau_1}$, $Y_2 = a^i\ \varepsilon_i|_{\tau_2}$. This suggests the following

Definition 15 *We say that $Y \in \sec T^{*}_{\sigma}M$ is transported without rotation* (*Fermi transported*) *if and only if $\mathcal{F}_{\epsilon_{\mathbf{0}}}Y = 0$.*

In that case we have

$$\begin{aligned} \boldsymbol{D}_{\epsilon_{\mathbf{0}}}Y = \varepsilon_{\mathbf{0}} \cdot \boldsymbol{\partial}Y \equiv \frac{D}{d\tau}Y &= (Y \cdot \varepsilon_{\mathbf{0}})a - (a \cdot Y)\varepsilon_{\mathbf{0}} \\ &= Y \lrcorner (\varepsilon_{\mathbf{0}} \wedge a) = (a \wedge \varepsilon_{\mathbf{0}}) \llcorner Y. \end{aligned} \tag{87}$$

[24]Recall the notations introduced in Chapter 4, where $\mathtt{g}$ denote the metric in the cotangent bundle.

D.1 Frenet Frames over σ and the Darboux Biform

Proposition 16 *If $\{\varepsilon_{\mathbf{a}}\}$ is a comoving coframe over σ, then there exists a unique biform field Ω_{D} over σ, called the angular velocity (Darboux biform) such that the $\varepsilon_{\mathbf{a}}$ satisfy the following system of differential equations*

$$\begin{aligned}\boldsymbol{D}_{\epsilon_{\mathbf{0}}}\varepsilon_{\mathbf{a}} &= \Omega_{\mathrm{D}}\llcorner\varepsilon_{\mathbf{a}},\\ \Omega_{\mathrm{D}} &= \frac{1}{2}\omega_{\mathbf{ab}}\varepsilon^{\mathbf{a}}\wedge\varepsilon^{\mathbf{b}} = -\frac{1}{2}(\boldsymbol{D}_{\epsilon_{\mathbf{0}}}\varepsilon_{\mathbf{b}})\wedge\varepsilon^{\mathbf{b}}.\end{aligned} \tag{88}$$

Proof. The proof is trivial and we also have the following ∎

Corollary 17 *If the comoving coframe $\{\varepsilon_{\mathbf{a}}\}$ is Fermi transported then the angular velocity is Ω_F,*

$$\Omega_F = a\wedge\varepsilon_{\mathbf{0}}. \tag{89}$$

D.2 Physical Meaning of Fermi Transport

Suppose that a comoving coframe $\{\varepsilon_{\mathbf{a}}\}$ is Fermi transported along a timelike curve σ, 'materialized' by some particle. The physical meaning associated to Fermi transport is that the spatial axis of the tetrad $\mathtt{g}(\varepsilon_i,) = \epsilon_i, i = 1,2,3$ are to be associated to the orthogonal spatial directions of three 'small' gyroscopes carried along σ.

Definition 18 *A Frenet coframe $\{f_{\mathbf{a}}\}$ over σ is a moving coframe over σ such that $f_{\mathbf{0}} = \boldsymbol{g}(\sigma_*,) = \boldsymbol{g}(\epsilon_{\mathbf{0}},) = \varepsilon_{\mathbf{0}}$ and*

$$\begin{aligned}\boldsymbol{D}_{\epsilon_{\mathbf{0}}}f_{\mathbf{a}} &= \Omega_{\mathrm{D}}\llcorner f_{\mathbf{a}},\\ \Omega_{\mathrm{D}} &= \kappa_0 f^{\mathbf{1}}\wedge f^{\mathbf{0}} + \kappa_1 f^{\mathbf{2}}\wedge f^{\mathbf{1}} + \kappa_2 f^{\mathbf{3}}\wedge f^{\mathbf{2}},\end{aligned} \tag{90}$$

where $\kappa_i, i = 0,1,2$ is the i-curvature, which is the projection of Ω_{D} in the $f^{\mathbf{i+1}}\wedge f^{I}$ plane.

Definition 19 *We say that a 1-form field Y over σ is rotating if and only if it is rotating in relation to gyroscopes axis, i.e., if $\mathcal{F}_{\epsilon_{\mathbf{0}}}Y \neq 0$.*

D.3 Rotation 2-form, the Pauli-Lubanski Spin 1-Form and Classical Spinning Particles

From Eq.(90) taking into account that $a = \boldsymbol{D}_{\epsilon_{\mathbf{0}}}f_{\mathbf{0}} = \kappa_{\mathbf{0}}f^{\mathbf{1}}$ and Eq.(89) we can write

$$\Omega_{\mathrm{D}} = a\wedge f_{\mathbf{0}} + \Omega_{\mathbf{S}}. \tag{91}$$

We now show that the 2-form $\Omega_{\mathbf{S}}$ over σ is directly related (a dimensional factor apart) with the spin 2-form of a *classical spinning particle*. More, we show now that the Hodge dual of $\Omega_{\mathbf{S}}$ is associated with the Pauli-Lubanski 1-form. To have a notation as closely as possible the usual ones of physical textbooks, let us put $f_{\mathbf{0}} = v$.

Call $\star\Omega_{\mathbf{S}}$ the Hodge dual of $\Omega_{\mathbf{S}}$. It is the 2-form over σ given by

$$\star\,\Omega_{\mathbf{S}} = -\Omega_{\mathbf{S}} f^{\mathbf{5}},\ f^{\mathbf{5}} = f^{\mathbf{0}} f^{\mathbf{1}} f^{\mathbf{2}} f^{\mathbf{3}}. \tag{92}$$

Define the *rotation* 1-form $\mathbf{S}$ over σ by

$$\mathbf{S} = -\star\,\Omega_{\mathbf{S}} \llcorner v. \tag{93}$$

Since $\Omega_{\mathbf{S}} \llcorner v = 0$, we have immediately that

$$\mathbf{S}\cdot v = -\star\,\Omega_{\mathbf{S}} \llcorner (v \wedge v) = 0. \tag{94}$$

Now, since $\boldsymbol{D}_{\epsilon_{\mathbf{0}}}(\mathbf{S}\cdot v) = 0$ we have that $(\boldsymbol{D}_{\epsilon_{\mathbf{0}}}\mathbf{S})\cdot a = -\mathbf{S}\cdot a$ and then

$$\boldsymbol{D}_{\epsilon_{\mathbf{0}}}\mathbf{S} = -\mathbf{S} \lrcorner (a \wedge v), \tag{95}$$

and it follows that

$$\mathcal{F}_{\epsilon_{\mathbf{0}}}\mathbf{S} = 0. \tag{96}$$

It is intuitively clear that we must associate $\mathbf{S}$ with the spin of a classical spinning particle which follows the worldline σ. And, indeed, we define the *Pauli-Lubanski* spin 1-form by

$$W = k\hbar\mathbf{S},\ \hbar = 1, \tag{97}$$

where $k > 0$ is a real constant and $\hbar$ is Planck constant, which is equal to 1 in the natural system of units used in this paper. We recall that as it is well-known $\boldsymbol{D}_{\epsilon_{\mathbf{0}}}W = -W \lrcorner (a \wedge v)$ is the equation of motion of the intrinsic spin of a classical spinning particle which is being accelerated by a force producing no torque .

References

[1] Boudet, R., *Quantum Mechanics in the Geometry of Space-Time. Elementary Theory*, Springer Briefs in Physics, Springer, Heidelberg, 2011.

[2] Bohm, D. and Hiley, B. J., *The Undivided Universe - An Ontological Interpretation of Quantum Theory*, Routledge, London (1993).

[3] Chen, P. and Kleinert, H., *Bohm Trajectories as Approximations to Properly Fluctuating Quantum Trajectories* [arXiv:1308.5021v1 [quant-ph]]

[4] Chen, P. and Kleinert, H., Deficiencies of Bohmian Trajectories in View of Basic Quantum Principles, *Elect. J. Theor .Phys.***13**,1-12 (2016). http://www.ejtp.com/articles/ejtpv13i35.pdf

[5] Crumeyrolle, A., *Orthogonal and Symplectic Clifford Algebras*, Kluwer Acad. Publ., Dordrecht, 1990.

[6] Daviau, C., Solutions of the Dirac Equation and of a Nonlinear Dirac Equation for the Hydrogen Atom, *Adv. Appl. Clifford Algebras*, **7(S)** 175-194 (1997).

[7] de Broglie, L., *Non-Linear Wave Mechanics. A Causal Interpretation*, Elsevier Publ. Co., Amsterdam, 1960.

[8] de Gosson, M., Hiley, B., Cohen, E., *Observing Quantum Trajectories: From Mott's Problem to Quantum Zeno Effect and Back*. [arXiv:1606.060651v1 {quant-ph]]

[9] Doran, C. and Lasenby, A., *Geometric Algebra for Physicists*, Cambridge University Press, Cambridge, 2003.

[10] Englert, B-G, Scully, M. O., Süssmann, G. and Walther, H.,Surrealistic Bohm Trajectories, *Z. Naturforsch.* **47A**, 1175- (1992)

[11] Geroch, R. Spinor Structure of Space-Times in General Relativity I, *J. Math. Phys.* **9**, 1739-1744 (1968).

[12] Geroch, R. Spinor Structure of Space-Times in General Relativity. II, *J. Math. Phys.* **11**, 343-348 (1970).

[13] Giéres, F., *Supersymmetries et Mathématiques*. Preprint LYCEN 9419, hep-th/940501.

[14] Giglio, J.F.T. and Rodrigues, W. A. Jr., Locally Inertial Reference Frames in Lorentzian and Riemann- Cartan Spacetimes, *Annalen der Physik.* **502**, 302-310 (2012).

[15] Hestenes, D., *Space-Time Algebra* (second revised edition), Birkhäuser, Basel, 2015.

[16] Hestenes, D., Local Observables in Dirqc Theory, *J. Math. Phys.* **14**, 893-905 (1973).

[17] Hestenes, D.,, Observables, Operators and Complex Numbers in Classical and Quantum Physics, J. Math. Phys. 16, 556-572 (1975)

[18] Hestenes, D., Real Dirac Theory, in Keller, J. and Oziewicz, Z. (eds.) Proceedings of the International Conference on Theory of the Electron (Mexico City, September 24-27,1995), *Adv. Applied Clifford Algbras* **7**(S), 97-144 (1997).

[19] Hiley, B. J. and Callaghan, R. E., Delayed Choice Experiments and the Bohm Approach. *Phys. Scr.* **74**, (2006) 336-348. arXiv 1602.06100.

[20] Hiley, B. J. and Callaghan, R. E., Clifford Algebras and the Dirac-Bohm Quantum Hamilton-Jacobi Equation, *Foundations of Physics* **42** (2012) 192-208.

[21] Hiley, B. J. and Callaghan, R. E., *The Clifford Algebra Approach to Quantum Mechanics B: The Dirac Particle and its relation to the Bohm Approach*, (2010) arXiv: 1011.4033.

[22] Holland, P. R., *The Quantum Theory of Motion*, Cambridge, University Press, London 1995.

[23] Jones, E., Bach, R., and Batelaan, H. Path integrals, Matter Waves, and the Double Slit, *Eur. J. Phys.* **36**, 065048 (2015) http://digitalcommons.unl.edu/cgi/viewcontent.cgi?article=1001&context=physicsbatelaan

[24] Landau, L. D. and Lifshitz, E. M., *The Classical Theory of Fields*, fourth revised English edition, Pergamon Press, New York, 1975.

[25] Lawson, H. Blaine, Jr. and Michelson, M. L., *Spin Geometry*, Princeton University Press, Princeton, 1989.

[26] Lounesto, P., *Clifford Algebras and Spinors*, Cambridge Univ. Press, Cambridge, 1997.

[27] Mahler, D. H., Rozema, L., Fisher, K.,Vermeyden,L., Resch, K. J., Wiseman, H. M. and Steinberg A., Experimental Nonlocal and Surreal Bohmian Trajectories, *Sci. Adv.* **2** ,e1501466 (2016). http://advances.sciencemag.org/content/advances/2/2/e1501466.full.pdf

[28] Oliveira, E. Capelas de, and Rodrigues, W. A. Jr., and Vaz, J. Jr., *Elko Spinor Fields and Massive Magnetic Like Monopoles, Int. J. Theor. Physics (* 2014) [arXiv:1306.4645 [math-ph]].

[29] Rodrigues, W. A. Jr., and Vaz J. Jr., Subluminal and Superluminal Solutions in Vacuum of the Maxwell Equations and the Massless Dirac Equation. Talk presented at the International Conference on the Theory of the Electron, Mexico City, 1995, *Adv. Appl. Clifford Algebras* **7** (Sup.), 453-462 (1997).

[30] Rodrigues, W. A. Jr., and Lu, J. Y., On the Existence of Undistorted Progressive Waves (UPWs) of Arbitrary Speeds $0 \leq v < \infty$ in Nature, *Found. Phys.* **27**, 435–508 (1997).

[31] Mosna, R. A. and Rodrigues, W. A. Jr., The Bundles of Algebraic and Dirac-Hestenes Spinor Fields, *J. Math. Phys.* **45**, 2945-2966 (2004).

[32] Rodrigues, W. A. Jr., Algebraic and Dirac-Hestenes Spinors and Spinor Fields, *J. Math. Phys.* **45**, 2908-2994 (2004).

[33] Rodrigues, W. A. Jr., Vaz, J. Jr. and Pavsic, M, The Clifford Bundle and the Dynamics of the Superparticle, *Banach Center Publications. Polish Acad. Sci.* **37**, 295-314 (1996).

[34] Rodrigues,W. A..Jr. .and Capelas de Oliveira,E., *The Many Faces of Maxwell, Dirac and Einstein Equations. A Clifford Bundle Approach* (second revised and enlarged edition),Springer,Heidelberg, 2016.

[35] Sachs, R. K., and Wu, H., *General Relativity for Mathematicians*, Springer-Verlag, New York, 1977.

[36] Sawant, R., Samuel, J., Sinha, A., Sinha, S.and Sinha, U., *Non-Classical Paths in Interference Experiments*, [arXiv:1308.2022v2 [quant-ph]].

[37] Vaz, J. Jr. and da Rocha, R., *An Introduction to Clifford Algebras and Spinors*, Oxford Univ. Press, Oxford, 2016.

[38] Zeni, J. R. R. and Rodrigues, W. A., Jr., A Thoughtful Study of Lorentz Transformations by Clifford Algebras, *Int. J. Mod. Phys. A* **7**, 1793-1817 (1992).